0.9 Solving Equations by Graphing

Let's consider an equation we can solve algebraically and see the connection between the solution of the equation and what we see on the graph. This gives us the technique we need to solve an equation graphically when we cannot solve it algebraically.

Work through these directions to understand the *technique.* We will then apply this method to an equation we cannot solve algebraically, and see the specific steps we need to use in order to solve the equation graphically.

Example

Using your algebra skills solve this equation, showing your work below.

$$\frac{1}{2}x - 3 = 2x - 1$$

Did you find $x = -\frac{4}{3}$ as the solution? If not, check your work again.

Now, let's solve this same equation graphically. We can use two different methods when solving an equation graphically. One method utilizes $x-$intercepts and the other depends on points of intersection of the graphs. You should learn both methods, and then choose the one you prefer for the particular problem at the time. Looking for $x-$intercepts will be presented first since it is often easier than finding the points of intersection, and it seems to be less error-prone.

0.9.1 $x-$intercept Method

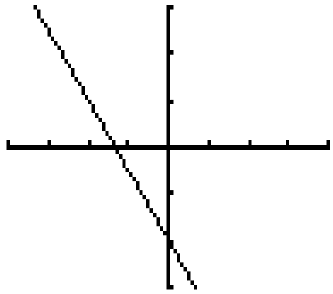
- Set the equation equal to zero by rearranging the terms.

  $$\frac{1}{2}x - 3 = 2x - 1 \quad \text{becomes} \quad \frac{1}{2}x - 3 - 2x + 1 = 0$$

  We can simplify the expression $\frac{1}{2}x - 3 - 2x + 1$, but usually we want to let the calculator do the work for us, especially if the expression is complicated. So let's do nothing more than rewrite the equation as $\frac{1}{2}x - 2x - 2 = 0$. Do you agree that to solve the original equation $\frac{1}{2}x - 3 = 2x - 1$ that we can solve the equivalent equation $\frac{1}{2}x - 2x - 2 = 0$? The solution for either one of these equations will also be the solution for the other equation.

- To solve $\frac{1}{2}x - 2x - 2 = 0$ by using $x-$intercepts, we graph $y =$ nonzero side of the equation.

  Graphing $y = \frac{1}{2}x - 2x - 2$ with a window of $[-4, 4]$ by $[-3, 3]$, we have



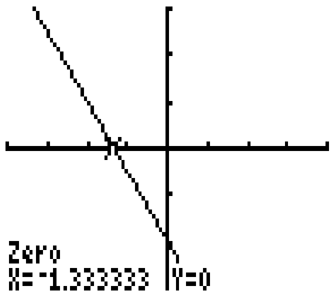
- Find the $x-$intercept:

  The $x-$intercept is also called the *zero* or the *root* of the function.

  Specific directions for finding the $x-$intercept on each calculator will be given in the next example.

  For now, suppose we have found it as shown.

  The $x-$intercept of -1.333333 is the same as our solution by hand of $x = -\frac{4}{3}$.



  Why will the $x-$intercept give us the solution to our equation?

  Given the equation $\frac{1}{2}x - 2x - 2 = 0$ we want the value of x which makes the equation equal to zero.

  Given the function $y = \frac{1}{2}x - 2x - 2$, to find algebraically the x-intercept, we set $y = 0$ and solve for x. Thus, we have $0 = \frac{1}{2}x - 2x - 2$ but this is the equation we wish to solve. Therefore, on the graph, the x-intercept gives the value of x which makes $y = 0$, and this $x-$value is also the solution to our equation.

Now let's solve $x^3 - 3x^2 + x + 1.5 = -2x^2 + 2x + 1$. This equation cannot be solved algebraically with our current techniques. Using the $x-$intercept method, we need to:

1. Set the equation equal to 0 by rearranging the terms:
   $$x^3 - 3x^2 + x + 1.5 + 2x^2 - 2x - 1 = 0 \quad \text{or} \quad x^3 - x^2 - x + .5 = 0$$
2. Graph $y = x^3 - x^2 - x + .5$
3. Find the $x-$intercepts. These $x-$values are the solutions to the equation
   $$x^3 - x^2 - x + .5 = 0$$

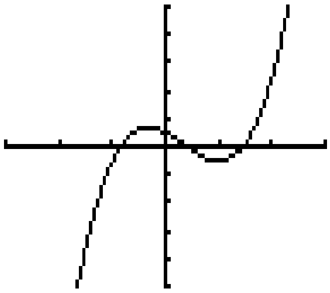
TI-83 (see page 126), TI-89 (see page 130), TI-86 (see page 133).

Choose the particular directions for your calculator. Try these instructions on your calculator as you read. Ask for help if you have difficulties.

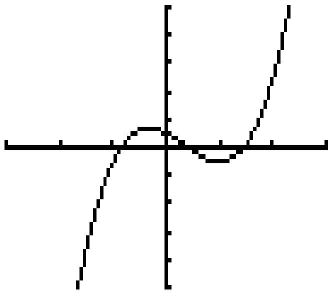
TI-83: $x-$intercept Method

To solve $x^3 - 3x^2 + x + 1.5 = -2x^2 + 2x + 1$,
we need to graph $y = x^3 - x^2 - x + .5$.

Using a window of $[-3, 3]$ by $[-5, 5]$,
we have this graph.



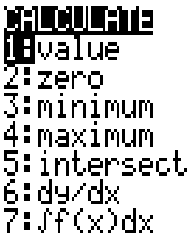
Note, instead of simplifying the equation, if we graphed
$y = x^3 - 3x^2 + x + 1.5 + 2x^2 - 2x - 1$,
we have the same graph.



To find the three $x-$intercepts, we need to use the **zero function**.

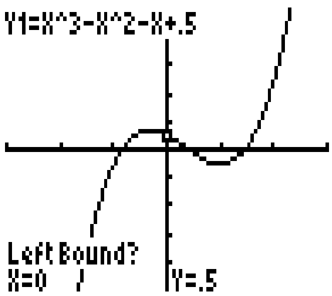
To find the leftmost $x-$intercept

Press (2nd) (Calc)
(Calc is the second function for (Trace)).



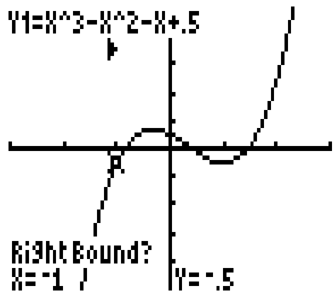
Press 2 to select the *zero* feature
and we immediately see the graph again.

(Note, the function definition for y_1 is shown in the upper left corner. If you do not see this definition and wish to, refer to section 0.8.2.)



We have to select an interval which "brackets" the $x-$intercept we need. We are prompted to indicate a *Left Bound*. This means we must choose an $x-$value to the *left* of the $x-$intercept. We may use the arrow keys to move the cursor along the graph to the left of the $x-$intercept, or we may type in an $x-$value which is left of the $x-$intercept.

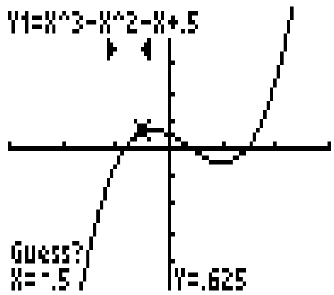
Suppose we type -1 and press Enter.



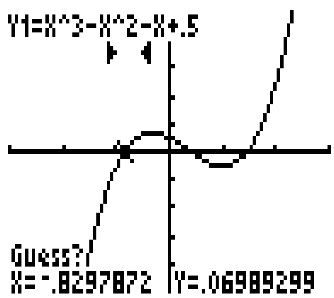
Notice the arrow on the graph above $x = -1$ pointing to the right. This indicates the left endpoint of the interval we are considering for the $x-$intercept.

We are prompted for the *Right Bound*. Follow the same process - use the arrow keys to move the cursor along the graph to the *right* of the $x-$intercept, or type in an $x-$value which is to the right of the desired $x-$intercept.

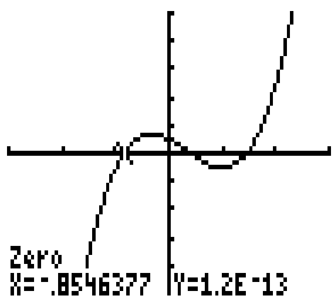
Let's type $-.5$ and press Enter:
See the left and right arrows on the graph which bracket the $x-$value for the $x-$intercept? If the $x-$intercept is not between these arrows, press Clear and repeat this process.



The *Guess?* prompt asks us to make a guess for the $x-$intercept. Move the cursor with the left/right arrow keys closer to the appropriate spot for the $x-$intercept if you wish.



After pressing Enter, we have the exact $x-$intercept, (or the *zero* of the function), of $x = -.85$, rounding to two decimal places.

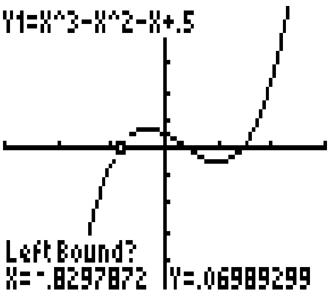


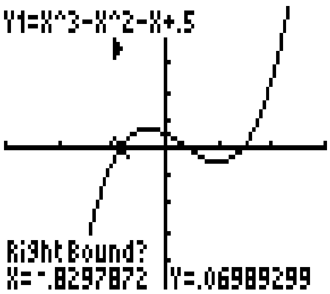
Notice the $y-$coordinate for this $x-$intercept is shown as $y =$ `-1.2E-13`. This number is extremely small. The `E-13` means we move the decimal point 13 places to the left, so `-1.2E-13` represents $-.00000000000012$. We accept this as $y = 0$. It is the best the calculator can do. (Besides, rounding even to the 12^{th} decimal place, we get 0.)

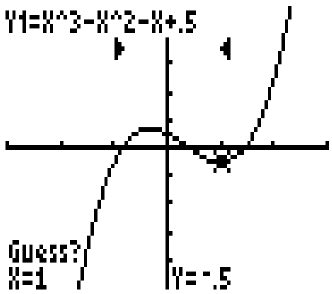
Now repeat the process for the next two $x-$intercepts.

Middle $x-$intercept:

Press 2nd Calc to start the process again for finding a *zero*. Select `2:zero` (left picture). To set the *Left Bound*, the cursor is currently on the left of our middle $x-$intercept, so we just need to press Enter (middle picture). To set the *Right Bound*, let's enter 1 and press Enter (right picture).

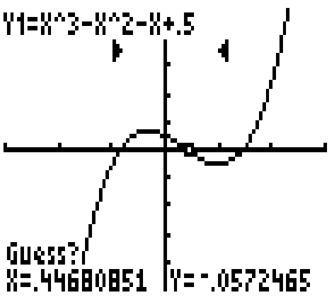


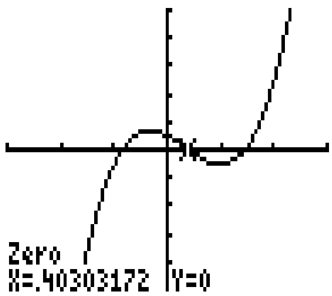




Notice the arrows across the top indicate an appropriate interval for our middle $x-$intercept.

For the *Guess?*, move the cursor closer to the middle $x-$intercept if you wish (left picture). Press Enter and we have the *zero* of $x = .40$, rounding to two decimal places (right picture).

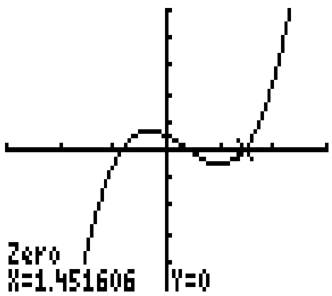




Rightmost $x-$intercept

Find the third $x-$intercept.

Did you find $x = 1.451606$, or $x = 1.45$ to two decimal places?



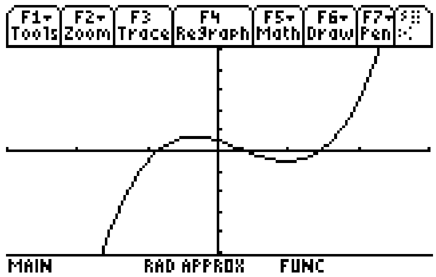
If not, try one of the first two intercepts again, then repeat this one. If it still does not work for you, ask for some help.

Continue reading on page 136 for a summary of this technique, followed by the second method for solving equations by using points of intersection.

TI-89: $x-$intercept Method

To solve $\quad x^3 - 3x^2 + x + 1.5 = -2x^2 + 2x + 1$,
we need to graph $y = x^3 - x^2 - x + .5$.

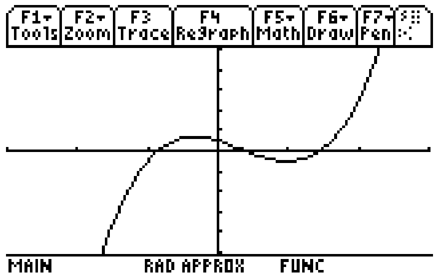
Using a window of $[-3, 3]$ by $[-5, 5]$,
we have this graph.



Note, instead of simplifying the equation, if we graphed

$$y = x^3 - 3x^2 + x + 1.5 + 2x^2 - 2x - 1$$

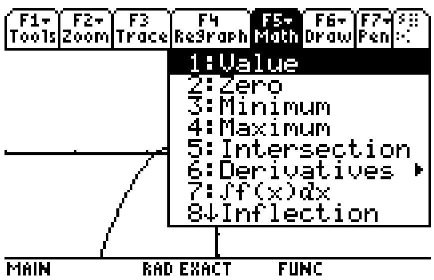
we have the same graph.

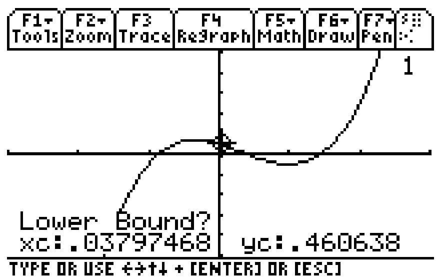


To find the three $x-$intercepts, we need to use the **zero function**.

<u>To find leftmost $x-$intercept</u>

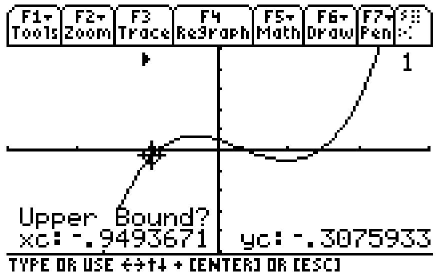
Press F5 Math (left picture). Then press 2 to select the *zero* feature and we see the graph again (right picture). (The 1 in the upper right corner is indicating we are working with the graph for function y_1.)





We have to select an interval which "brackets" the $x-$intercept we need. We are prompted to select a *Lower Bound*. This means we must choose an $x-$value to the *left* of the $x-$intercept. We may use the arrow keys to move the cursor along the graph to the left of the $x-$intercept, or we may type in an $x-$value which is left of the $x-$intercept.

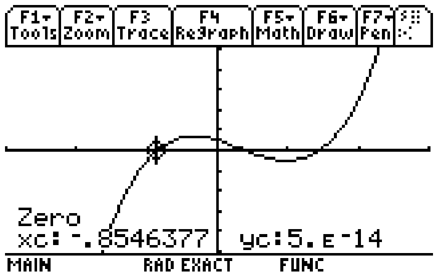
Suppose we type −1 and press (Enter).
(The calculator actually shows $x = -.9493671$.
You may get a slightly different value.)



Notice the arrow above $x = -1$ showing the left endpoint of the interval we are considering for the $x-$intercept.

We are prompted for the *Upper Bound*. Follow the same process - use the arrow keys to move the cursor along the graph to the *right* of the $x-$intercept, or type in an $x-$value from the keyboard which is to the right of the desired $x-$intercept.

Let's type −.5. Pressing (Enter) we have the exact $x-$intercept (or the *zero* of the function) of $x = -.85$, rounding to two decimal places.

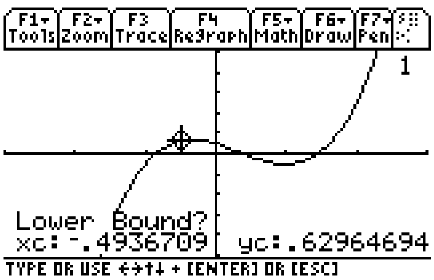


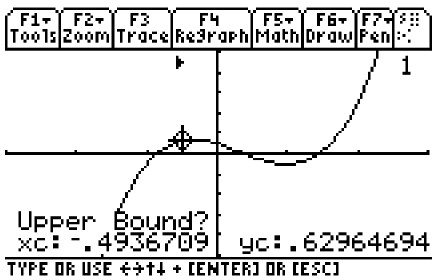
Notice the $y-$coordinate for this $x-$intercept is shown as $y =$ `-5.E-14`.
This number is extremely small. The `E-14` means we move the decimal point 14 places to the left, so `-5.E-14` represents −.00000000000005. We accept this as $y = 0$. It is the best the calculator can do. (Besides, rounding even to the 13^{th} decimal place, we get 0.)

Now repeat the process for the next two $x-$intercepts.

<u>Middle $x-$intercept</u>

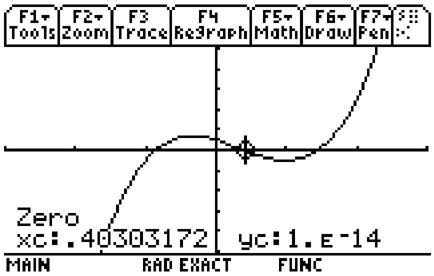
Press (F5 Math) to start the process again for finding a zero. Select `2:Zero` (left picture). To set the *Lower Bound*, since the cursor is currently on the left of our middle $x-$intercept, we can press (Enter) (right picture).





To set the *Upper Bound*, let's type 1, press (Enter) :

We have the *zero* of $x = .40$, rounding to two decimal places.

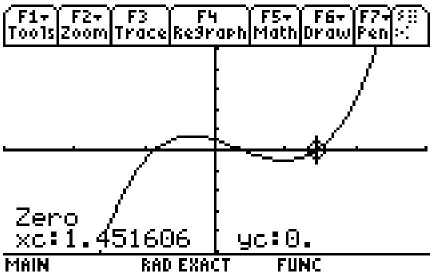


Notice, again the $y-$value for the $x-$intercept is not exactly zero, but an extremely small value, which we accept as zero.

Rightmost $x-$intercept

Find the third $x-$intercept.

Did you find $x = 1.451606$, or $x = 1.45$ to two decimal places?



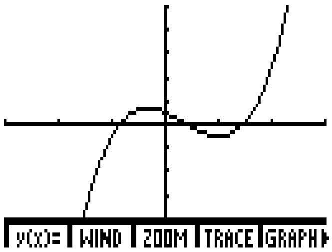
If not, try one of the first two intercepts again, then repeat this one. If it still does not work for you, ask for some help.

Continue reading on page 136 for a summary of this technique, followed by the second method for solving equations by using points of intersection.

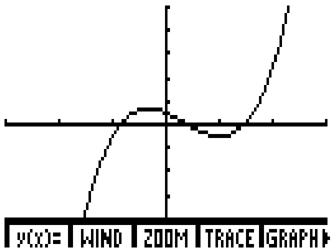
TI-86: x−intercept Method

To solve $x^3 - 3x^2 + x + 1.5 = -2x^2 + 2x + 1$,
we need to graph $y = x^3 - x^2 - x + .5$.

Using a window of $[-3, 3]$ by $[-5, 5]$,
we have this graph.



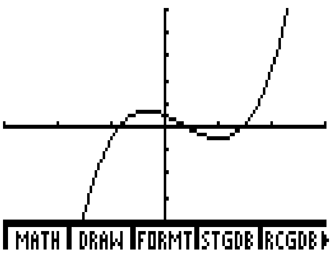
Note, instead of simplifying the equation, if we graphed $y = x^3 - 3x^2 + x + 1.5 + 2x^2 - 2x - 1$, we have the same graph.

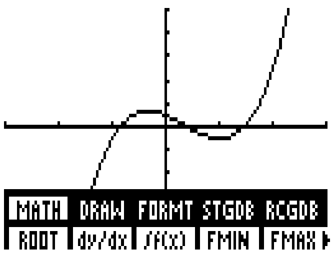


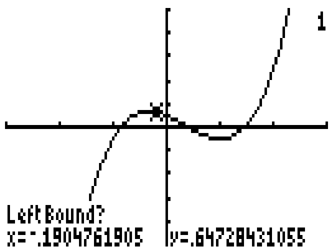
To find the three x−intercepts, we need to use the **root function**.

To find the leftmost x−intercept

Press More to see additional options for the `Graph` feature (left picture). Press F1 Math (middle picture). Press F1 Root (right picture).





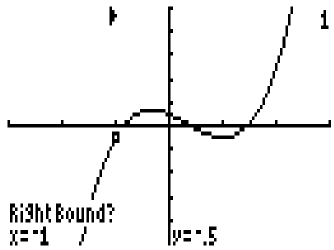


We have to select an interval which "brackets" the x−intercept we need. We are prompted to indicate a *Left Bound*. This means we must choose an x−value to the *left* of the x−intercept. We may use the arrow keys to move the cursor along the graph to the left of the x−intercept, or we may type in an x−value which is left of the x−intercept.

The 1 in the upper right corner indicates we are working with the graph for function y_1.

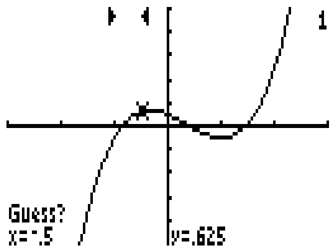
Suppose we type -1 and press Enter.

Notice the arrow on the graph above $x = -1$ pointing to the right. This indicates the left endpoint of the interval we are considering for the $x-$intercept.



We are prompted for the *Right Bound.* Follow the same process - use the arrow keys to move the cursor along the graph to the *right* of the $x-$intercept, or type in an $x-$value which is to the right of the desired $x-$intercept.

Let's type $-.5$ and press Enter.
See the left and right arrows on the graph which bracket the $x-$value for the $x-$intercept? If the $x-$intercept is not between these arrows, press Clear, Graph and repeat this process.



The *Guess?* prompt asks us to make a guess for the $x-$intercept. Move the cursor with the left/right arrow keys closer to the appropriate spot for the $x-$intercept if you wish.

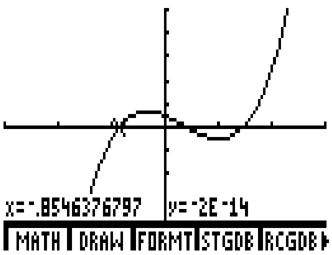
After pressing Enter, we have the exact $x-$intercept, (or the *root* of the function), of $x = -.85$, rounding to two decimal places.

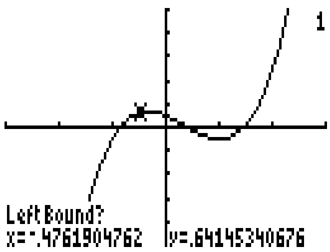
Notice the $y-$coordinate for this $x-$intercept is $y =$ `-2E-14`. This number is extremely small. The `E-14` means we move the decimal point 14 places to the left, so `-2E-14` represents $-.00000000000002$. We accept this as $y = 0$. It is the best the calculator can do. (Besides, rounding even to the 13^{th} decimal place, we get 0.)

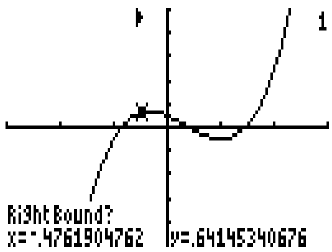
Now repeat the process for the next two $x-$intercepts.

Middle $x-$intercept

Press Exit to get back to the menu with the `Math` option and start the process over again (left picture). Press F1 Math, and F1 Root (middle picture). To set the *Left Bound*, we can press Enter since the cursor is currently on the left of the middle $x-$intercept (right picture).

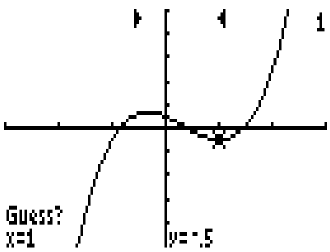


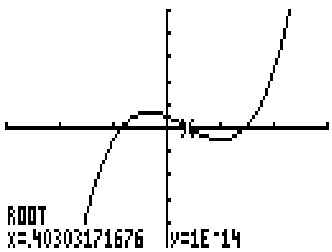




To set the *Right Bound*, let's type 1 and press Enter (left picture). Notice the arrows across the top indicate the interval we are using for our middle $x-$intercept.

For the *Guess?*, press Enter and we have the *root* of $x = .40$, rounding to two decimal places (right picture).



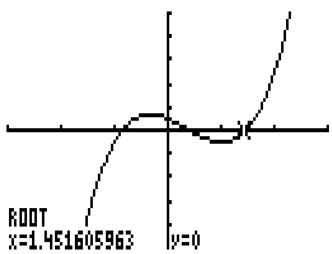


Again, the $y-$coordinate of the $x-$intercept is an extremely small number of `1E-14`, which we accept as zero.

Rightmost $x-$intercept

Find the third $x-$intercept.

Did you find $x = 1.451606$, or $x = 1.45$ to two decimal places?



If not, try one of the first two intercepts again, then repeat this one. If it still does not work for you, ask for some help.

Continue reading on page 136 for a summary of this technique, followed by the second method for solving equations by using points of intersection.

Conclusion: Solving Equations using $x-$intercepts

Given an equation $a = b$ to solve, where a and b are expressions, we can

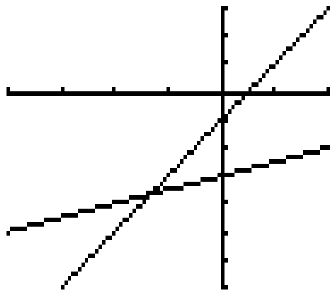
- Set the equation to 0 to get $a - b = 0$.
- Graph $y = a - b$.
- Find the $x-$intercept(s).
  The $x-$intercept(s) are the solutions to our original equation $a = b$.

0.9.2 Points of Intersection Method

The second method for solving equations graphically depends on finding the point(s) of intersection for the graphs.

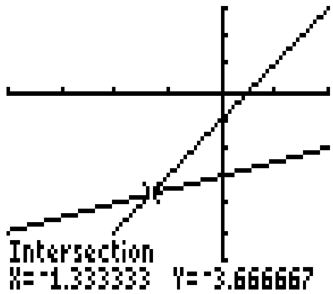
Returning to the equation we originally solved algebraically, let's see how the points of intersection method works. Recall, we found a solution of $x = -\frac{4}{3}$.

- To solve $\frac{1}{2}x - 3 = 2x - 1$ we express the left and right sides of the equation as two different functions: $y_1 = \frac{1}{2}x - 3$ and $y_2 = 2x - 1$
- Graphing both functions on the same coordinate system using a viewing window of $[-4, 2]$ by $[-5, 3]$, we have



These two functions are equal, $y_1 = y_2$, or $\frac{1}{2}x - 3 = 2x - 1$, when the two graphs intersect. We need to find the $x-$coordinate of the point of intersection. This $x-$value will be the solution to $y_1 = y_2$, or $\frac{1}{2}x - 3 = 2x - 1$.

- Finding point(s) of intersection
Using the `Intersection` feature to find the $x-$ and $y-$ coordinates of the point of intersection, we have (Directions given for `Intersection` in the next example.)



Now let's solve $x^3 - 3x^2 + x + 1.5 = -2x^2 + 2x + 1$ by using the points of intersection method.

Steps are given for each calculator in the following example. Try the directions for your calculator as you read. If you have any difficulties, ask for help.

TI-83 (see page 138), TI-89 (see page 141), TI-86 (see page 144).

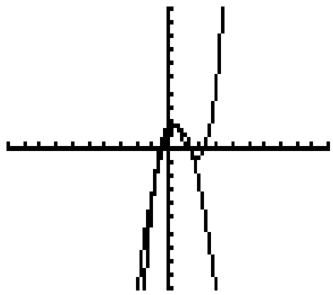
TI-83: Points of Intersection Method

To solve $x^3 - 3x^2 + x + 1.5 = -2x^2 + 2x + 1$ we need to graph

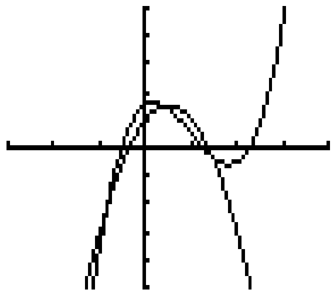
$$y_1 = x^3 - 3x^2 + x + 1.5 \quad \text{and} \quad y_2 = -2x^2 + 2x + 1$$

on the same coordinate system.

Using a standard viewing window of $[-10, 10]$ by $[-10, 10]$, we see the graph on the right.



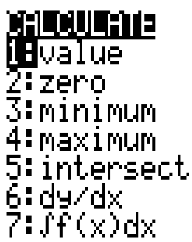
Adjusting the window to $[-3, 4]$ by $[-5, 5]$, we have a better view of the graphs.

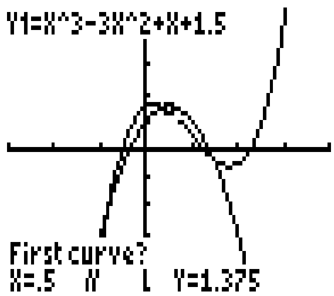


We can see that the curves intersect three times, close to $x = -1$, between 0 and 1, and between 1 and 2.

Thus, we need to use the `Intersect` feature of the calculator and find each point of intersection.

Press 2nd and Calc (left picture). Press 5 to select the *intersect* feature and return to the graph (right picture).



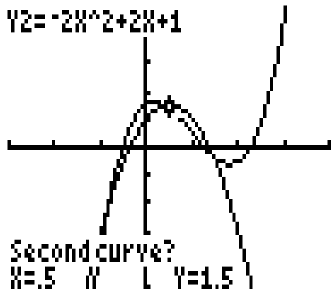


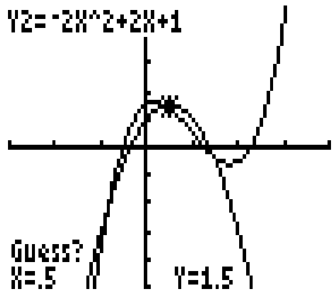
We are prompted for the *First curve.* Notice the upper left corner shows you the equation for the line the cursor is on. (If you do not see the formula for the function, refer to section 0.8.2 for directions.)

We must indicate which two graphs we want to work with when finding the point of intersection. (There could be more than two functions graphed, so which two are we working with this time?)

The cursor is currently on the first curve, y_1.

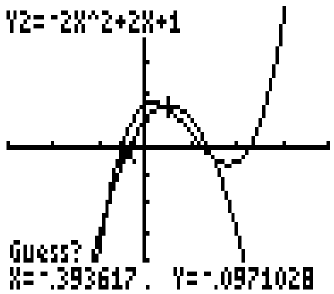
Press Enter to select this line as the first curve. The cursor immediately moves to the second curve (left picture). (In this picture, it is hard to see that the cursor is on the second line, but notice the function indicated at the top left is correct for the second curve.) We are prompted for the *Second curve*. Press Enter to indicate the cursor is on the second curve of interest (right picture).

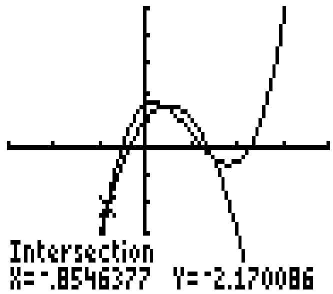




The *Guess?* prompt asks us to make a guess for the point of intersection.

For our guess, move the cursor closer to the leftmost point of intersection (left picture). Press Enter to find the actual intersection point (right picture).



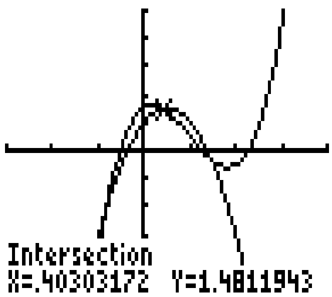


The curves intersect at $x = -.8546377$, $y = -2.170086$, or to two decimal places, $x = -.85$. Thus, $x = -.85$ is one solution to our equation.

Remember, we are using this technique to solve an equation for x; thus, the $x-$value is the only number we need for our solution.

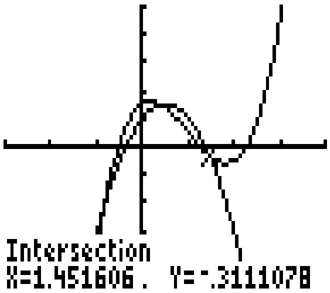
Press 2nd and Calc to start the process again. Find the next point of intersection.

Did you find $x = .40$ for the middle point of intersection?



Now find the rightmost point of intersection.

Did you find $x = 1.45$ for this point of intersection?



So the solutions to $x^3 - 3x^2 + x + 1.5 = -2x^2 + 2x + 1$ are $x = -.85,\ .40,$ and 1.45, when rounded to two decimal places. Notice that these three values agree with our previous solutions found when using the $x-$intercept method.

Continue reading on page 146 for a summary of this technique.

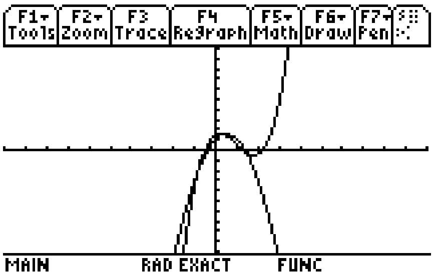
TI-89: Points of Intersection Method

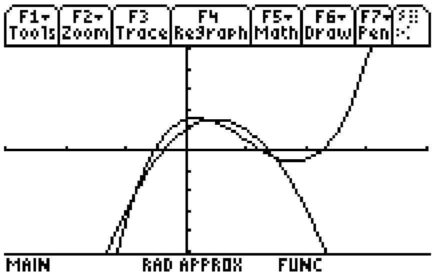
To solve $x^3 - 3x^2 + x + 1.5 = -2x^2 + 2x + 1$ we need to graph

$$y_1 = x^3 - 3x^2 + x + 1.5 \quad \text{and} \quad y_2 = -2x^2 + 2x + 1$$

on the same coordinate system.

Using a standard viewing window of $[-10, 10]$ by $[-10, 10]$, we see the graph on the left. Adjusting the window to $[-3, 4]$ by $[-5, 5]$, we have a better view of the graph (right picture).

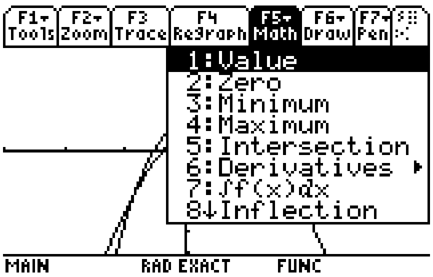


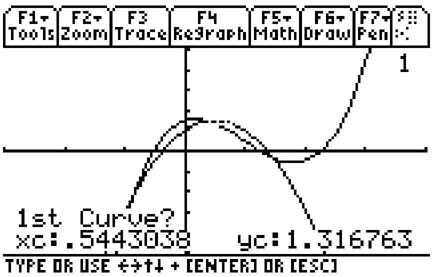


We see the curves intersect three times, close to $x = -1$, between 0 and 1, and between 1 and 2.

Thus, we need to use the `Intersect` feature of the calculator and find each point of intersection.

Press F5 Math (left picture). Press `5:Intersect` to return to the graph (right picture).

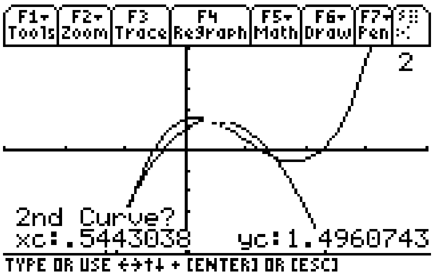


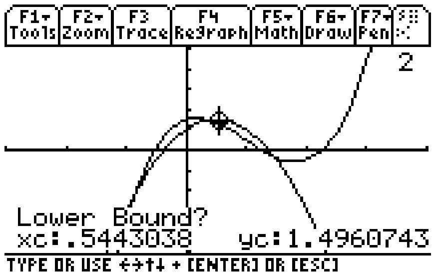


We are prompted for the *1st Curve*. Notice the upper right corner shows a 1 indicating the cursor is on equation y_1.

We must indicate which two graphs we want to work with when finding the point of intersection. (There could be more than two functions graphed, so which two are we working with this time?)

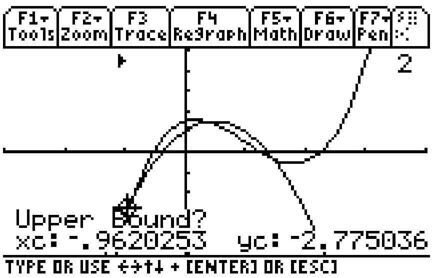
Press Enter to indicate the cursor is on the first curve of interest. The cursor immediately moves to the second curve (left picture). Notice the top right corner indicates the cursor is now on equation y_2. Press Enter again to indicate the cursor is on the second curve of interest (right picture).

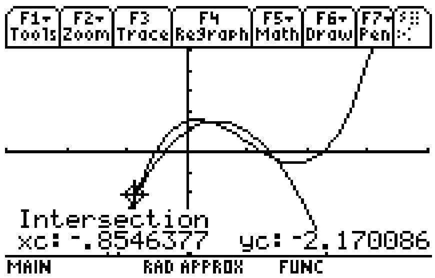




The prompt now requests a *Lower Bound*. Let's type -1, press Enter and we see the left picture below. (Note, the cursor is showing $x = -.9620253$. You may see something slightly different.) Notice the arrow on the left indicating the left endpoint for the interval of interest for this point of intersection.

The prompt now requests the *Upper Bound*. Either move the cursor to the *right* of the intersection point, or type in a numerical value for x. Suppose we enter 1. We immediately see the point of intersection (right picture).

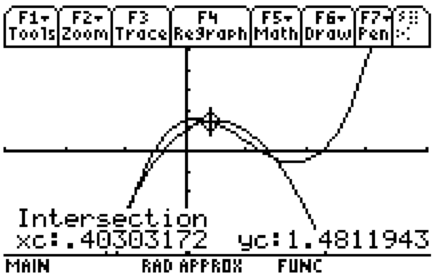




The curves intersect at $x = -.8546377$, $y = -2.170086$, or to two decimal places, $x = -.85$. Thus, $x = -.85$ is one solution to our equation.

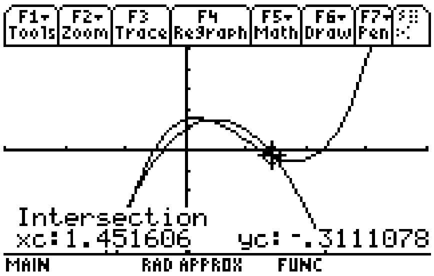
Press F5 Math to start the process again. Find the middle point of intersection.

Did you find $x = .40$?



Now find the rightmost point of intersection.

Did you find $x = 1.45$?



So the solutions to $x^3 - 3x^2 + x + 1.5 = -2x^2 + 2x + 1$ are $x = -.85,\ .40,$ and 1.45, when rounded to two decimal places. Notice that these three values agree with our previous solutions found when using the $x-$intercept method.

Continue reading on page 146 for a summary.

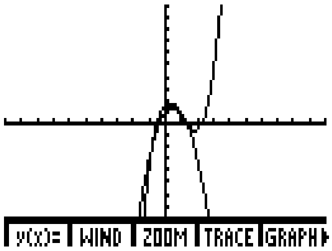
TI-86: Points of Intersection Method

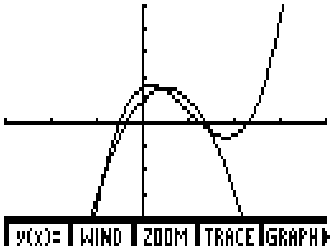
To solve $x^3 - 3x^2 + x + 1.5 = -2x^2 + 2x + 1$ we need to graph

$$y_1 = x^3 - 3x^2 + x + 1.5 \quad \text{and} \quad y_2 = -2x^2 + 2x + 1$$

on the same coordinate system.

Using a standard viewing window of $[-10, 10]$ by $[-10, 10]$, we see the graph on the left. Adjusting the window to $[-3, 4]$ by $[-5, 5]$, we have a better view of the graphs (right picture).

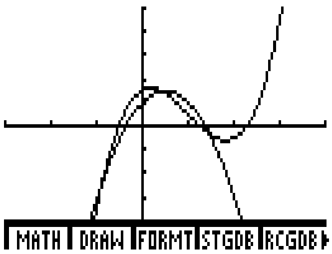


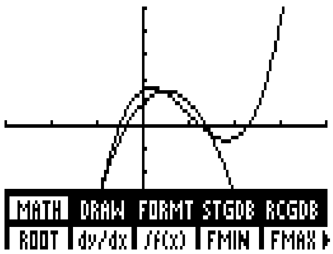


We can see that the curves intersect three times, close to $x = -1$, between 0 and 1, and between 1 and 2.

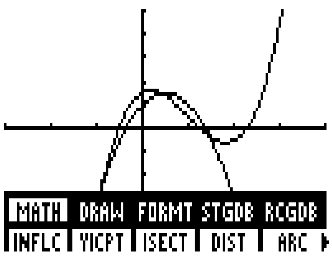
Thus, we need to use the `Intersect` feature of the calculator and find each point of intersection.

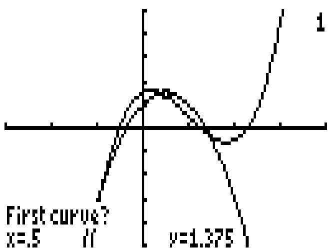
Press [More] to see additional options for the `Graph` feature (left picture). Press [F1 Math] (right picture).





Now press [More] again to see additional options (left picture). Press [F3 ISECT] for `Intersection` and to return to the graph (right picture).

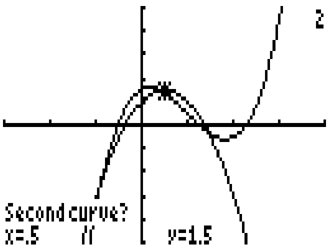


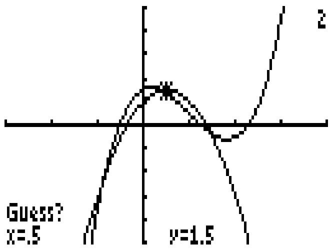


The prompt requests the *First curve*. Notice the upper right corner shows a 1 which indicates the cursor is on the line for y_1. (If you do not see this designation, refer to section 0.8.2 for directions.)

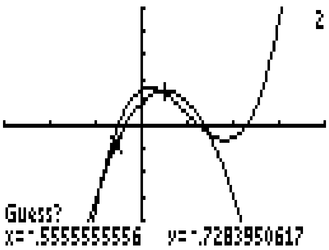
We must indicate which two graphs we want to work with when finding the point of intersection. (There could be more than two functions graphed, so which two are we working with this time?)

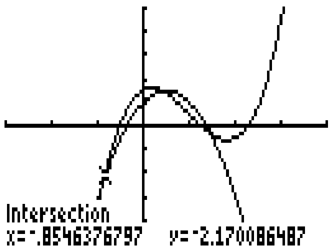
Press [Enter] to select this line as the first curve. The cursor moves immediately to the second curve (left picture). (In this picture, it is hard to see that the cursor is on the second line, but the function indicated in the upper right corner now shows a 2 for line y_2.) We are prompted for the *Second curve*. Press [Enter] to indicate the cursor is on the second curve of interest (right picture).





Now we see the *Guess?* prompt. Move the cursor with the left/right arrow keys closer to the point of intersection (left picture). Press [Enter] to see the answer (right picture).



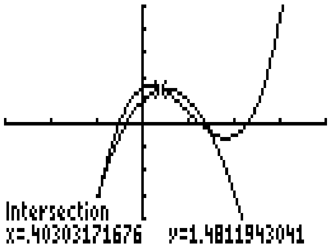


The cursor is now on the exact point of intersection. The $x-$ and $y-$coordinates of that point are shown as $x = -.8546376797$ and $y = -2.170086487$.

To two decimal places, the curves intersect at $x = -.85$. Thus, $x = -.85$ is one solution to our equation.

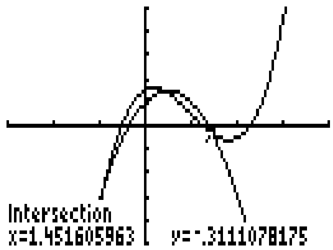
Press [Exit] to start process again. Press [F1 Math], and continue to find the middle point of intersection.

Did you find $x = .40303171676, y = 1.4811943041$, or using two decimal places, $x = .40$?



Now find the rightmost point of intersection.

Did you find $x = 1.451605963, y = -.3111078175$,
or using two decimal places, $x = 1.45$?



So the solutions to $x^3 - 3x^2 + x + 1.5 = -2x^2 + 2x + 1$ are $x = -.85,\ .40,$ and 1.45, when rounded to two decimal places. Notice that these three values agree with our previous solutions found when using the $x-$intercept method.

Conclusion: Solving Equations using Points of Intersection

Given an equation $a = b$, where a and b may be quite simple expressions or very complicated expressions, do the following.

- Graph $y_1 = a$ (the left-side of the equation) and
  graph $y_2 = b$ (the right-side of the equation).
- Find the $x-$coordinate of each point of intersection.
  This $x-$value is the solution to the equation $a = b$.

