Chapter 7

Analytic Trigonometry

Review sections as needed from Chapter 0, Basic Techniques, page 8. Remember to use radian measure for all graphing.

Use windows with $x-$values of a multiple of π when possible, such as $[-\pi, \pi]$. This will often make it easier to read the necessary information from the graph.

When graphing functions $\sec(x), \csc(x)$, or $\cot(x)$, do so by using the identities for these functions which involve $\sin(x)$ and $\cos(x)$. Do not just type in $\cot(x), \sec(x)$, or $\csc(x)$ if your calculator has that capability. The object here is to reinforce what we know about the trigonometric functions.

Refer to page 187 for an example of the work required on paper for all graded homework unless directed otherwise by your instructor.

7.1 Trigonometric Identities

Remember, to enter $\sin^2(x)$ on the calculator, we may enter $(\sin(x)) \wedge 2$, or $\sin(x) \wedge 2$

Exercises

1. Using $x-$values of $[-\pi, \pi]$, graph

   (a) $y = \sin^2(x)$

   (b) $y = \cos^2(x)$

   (c) Now graph both (a) and (b) on the same graph.

   (d) Explain why $\sin^2(x) + \cos^2(x) = 1$ by using your graphs from above.

2. Determine if the following equations are identities by graphing the left side and graphing the right side on your calculator on separate graphs. Are the graphs the same?

   For future reference, compare these equations and graphs carefully with the identity $\sin^2(x) + \cos^2(x) = 1$.

   (a) $\sin(x) + \cos(x) = 1$ (b) $\Big(\sin(x) + \cos(x)\Big)^2 = 1$

3. Determine if the following equations are identities by graphing the left side and graphing the right side on separate graphs. Are the graphs the same? (If so, you may wish to graph them on the same system to verify your answer.) Include any asymptotes when drawing the graph on paper.

   (a) $\tan^2(x) + 1 = \sec^2(x)$

   (b) $\tan(x) + 1 = \sec(x)$

4. Determine if this equation is an identity, by graphing the left side and graphing the right side on separate graphs. Are the graphs the same? (If so, you may wish to graph them on the same system to verify your answer.) Include any asymptotes when drawing the graph on paper. **If it is an identity, verify it algebraically.**

$$\tan(x) - 1 = \sec(x)\Big(\sin(x) - \cos(x)\Big)$$

5. Determine if the following equations are identities, by graphing the left side and graphing the right side on separate graphs. Are they the same? (If so, you may wish to graph them on the same system to verify your answer.) Include any asymptotes when drawing the graph on paper. **If it is an identity, verify it algebraically.**

   (a) $\cos(x) + \tan(x)\sin(x) = \sec(x)$

   (b) $\tan(x) + 1 = \sec(x)\Big(\sin(x) - \cos(x)\Big)$

6. Given $\dfrac{\cos x}{1 - \sin x} = \dfrac{\sin x - \csc x}{\cos x - \cot x}$

   (a) Verify this is an identity by graphing the left side of the equation and the right side of the equation on the same graph. Include any asymptotes when drawing the graph on paper.

   (b) Verify this identity algebraically.

7.2 Addition and Subtraction Formulas

Refer to page 187 for an example of the work required on paper for all graded homework unless directed otherwise by your instructor.

Exercises

1. Logan is convinced that $\sin(A + B) = \sin(A) + \sin(B)$. Tanner proposes a counterexample to prove Logan wrong. Tanner says $\sin(x + 3) \neq \sin(x) + \sin(3)$.

   (a) Graph $y = \sin(x + 3)$ and $y = \sin(x) + \sin(3)$ on your calculator. Who is correct? Explain why he is correct, by using transformations.

   (b) Graph $y = \sin(x + 3)$ and $y = \sin(x)\cos(3) + \cos(x)\sin(3)$. Are these two the same?

2. By graphing, decide if $y = \cos(x - 5)$ and $y = \cos(x)\cos(5) + \sin(x)\sin(5)$ are the same.

3. Simplify the trigonometric expression algebraically. Then graph the original expression on your calculator as y_1 and your simplified expression as y_2. If the graphs are not the same, find your mistake in either the algebra or in the graphing.

   (a) $\sin(x - \pi)$

   (b) $\cos\left(x + \dfrac{3\pi}{4}\right)$

   (c) $\sin(1.4x)\cos(0.9x) - \cos(1.4x)\sin(0.9x)$

4. Simplify the trigonometric expression algebraically. Then graph the original expression on your calculator as y_1 and your simplified expression as y_2. If the graphs are not the same, find your mistake in either the algebra or in the graphing.

   (a) $\sin\left(x - \dfrac{\pi}{2}\right)$

   (b) $\cos(1.4x)\cos(0.6x) - \sin(1.4x)\sin(0.6x)$

5. Given $\quad f(x) = -\frac{1}{2}\Big[\cos(x+\pi) + \cos(x-\pi)\Big]$

   (a) Graph $f(x)$ for two complete periods.

   (b) Describe this graph using a simpler function $g(x)$ in terms of sine or cosine.

   (c) Verify algebraically that $f(x) = g(x)$.

6. Given $\quad f(x) = \sin\left(x + \frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2} - x\right)$

   (a) Graph $f(x)$ for two complete periods.

   (b) Describe this graph using a simpler function $g(x)$ in terms of sine or cosine.

   (c) Verify algebraically that $f(x) = g(x)$.

7.3 Double-Angle and Half-Angle Formulas

Refer to page 187 for an example of the work required on paper for all graded homework unless directed otherwise by your instructor.

Exercises

1. (a) Graph, by hand, $y = \sin(2x)$ and $y = 2\sin(x)$. Are these two graphs the same; i.e., does $\sin(2x) = 2\sin(x)$?
   Explain your answer by stating what the 2 does in each case, in terms of amplitude and period, or in terms of transformations.

   (b) Graph on the calculator $y_1 = \sin(2x)$ and $y_2 = 2\sin(x)\cos(x)$.
   Are these two graphs the same? Does $\sin(2x) = 2\sin(x)\cos(x)$?

2. Graph these functions on the same graph. Are all the graphs equal? Are these expressions all different versions of the same thing?

   $y_1 = \cos(2x) \qquad y_2 = \cos^2(x) - \sin^2(x) \qquad y_3 = 1 - 2\sin^2(x) \qquad y_4 = 2\cos^2(x) - 1$

3. Given $f(x) = \dfrac{1 - 2\cos(2x)}{2\sin(x) - 1}$
   (a) Graph $f(x)$ for two complete periods.
   (b) Describe this graph using a simpler function $g(x)$ in terms of sine or cosine.
   (c) Verify algebraically that $f(x) = g(x)$.

4. Given $f(x) = \sin^2(x) - \cos^2(x)$
   (a) Graph $f(x)$ for two complete periods.
   (b) Describe this graph using a simpler function $g(x)$ in terms of sine or cosine.

5. Given $f(x) = 2\sin(x)\cos(x)$
   (a) Graph $f(x)$ for two complete periods.
   (b) Describe this graph using a simpler function $g(x)$ in terms of sine or cosine.

6. Given $f(x) = \sin\left(\dfrac{\pi}{2} - x\right)$
   (a) Graph $f(x)$ for two complete periods.
   (b) Describe this graph using a simpler function $g(x)$ in terms of sine or cosine.
   (c) Verify algebraically that $f(x) = g(x)$.

7. Given $f(x) = \cos^2(x) - \sin^2(x)$
   (a) Graph $f(x)$ for two complete periods.
   (b) Describe this graph using a simpler function $g(x)$ in terms of sine or cosine.

8. Given $f(x) = \sin(2x)\cos(2x)$
   (a) Graph $f(x)$ for two complete periods.
   (b) Describe this graph using a simpler function $g(x)$ in terms of sine or cosine.

9. Graph $f(x) = \sin(x) + \sin(2x)$ and $g(x) = \sin(3x)$.
   Does $f(x) = g(x)$; i.e. does $\sin(x) + \sin(2x) = \sin(3x)$?
   If yes, verify the identity algebraically.

10. Graph $f(x) = \cos(x) + \cos(3x)$ and $g(x) = 2\cos(2x)\cos(x)$.
    Does $f(x) = g(x)$; i.e. does $\cos(x) + \cos(3x) = 2\cos(2x)\cos(x)$?
    If yes, verify the identity algebraically.

7.4 Inverse Trigonometric Functions

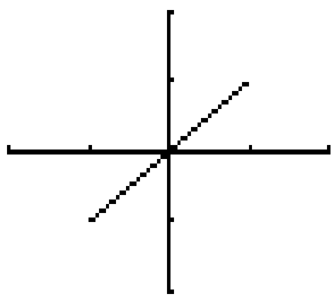
Refer to page 187 for an example of the work required on paper for all graded homework unless directed otherwise by your instructor.

<u>*Exercises*</u>

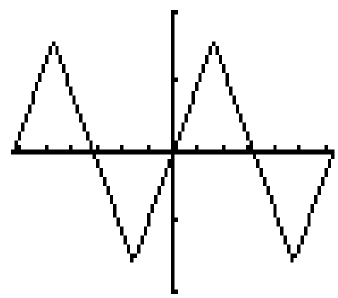
1. Using the Inverse Function Property, the functions $f(x) = \sin(\sin^{-1}(x))$ and $g(x) = \sin^{-1}(\sin(x))$ both simplify to x for *suitable values* of x, but these functions $f(x)$ and $g(x)$ are not equivalent to x for all real numbers x.

   Graphs of $f(x)$ and $g(x)$ are shown below. Explain why the graphs look the way they do, considering carefully the domain and range of $\sin^{-1}(x)$. Explain why the graphs of $f(x)$ and $g(x)$ are not the same.

   $f(x) = \sin(\sin^{-1}(x))$ on $[-2, 2], [-2, 2]$

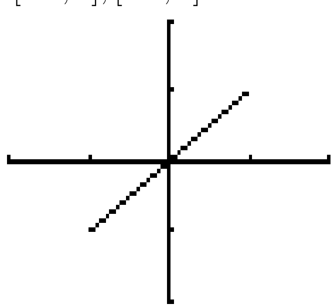
   $g(x) = \sin^{-1}(\sin(x))$ on $[-2\pi, 2\pi]$ by $[-2, 2]$

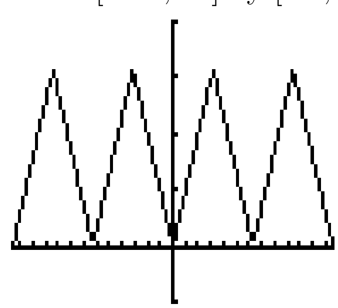
2. Using the Inverse Function Property, the functions $f(x) = \cos(\cos^{-1}(x))$ and $g(x) = \cos^{-1}(\cos(x))$ both simplify to x for *suitable values* of x, but these functions $f(x)$ and $g(x)$ are not equivalent to x for all real numbers x.

   Graphs of $f(x)$ and $g(x)$ are shown below. Explain why the graphs look the way they do, considering carefully the domain and range of $\cos^{-1}(x)$. Explain why the graphs of $f(x)$ and $g(x)$ are not the same.

   $f(x) = \cos(\cos^{-1}(x))$ on $[-2, 2], [-2, 2]$

   $g(x) = \cos^{-1}(\cos(x))$ on $[-4\pi, 4\pi]$ by $[-1, 4]$

3. Given $f(x) = \sin^{-1}(x)$ and $g(x) = \sin^{-1}(x) + 2$

   (a) Graph both functions on the same coordinate system, using what you know about the domain of $\sin^{-1}(x)$ to help select an appropriate window.

   (b) Explain how the graph of $g(x)$ is obtained from the graph of $f(x)$.

4. Given $f(x) = \cos^{-1}(x)$ and $g(x) = \cos^{-1}(x + 2)$

   (a) Graph both functions on the same coordinate system, using what you know about the domain of $\cos^{-1}(x)$ to help select an appropriate window.

   (b) Explain how the graph of $g(x)$ is obtained from the graph of $f(x)$.

5. Given $f(x) = \sin^{-1}(x)$ and $g(x) = -\sin^{-1}(x - 3)$

   (a) Graph both functions on the same coordinate system, using what you know about the domain of $\sin^{-1}(x)$ to help select an appropriate window.

   (b) Explain how the graph of $g(x)$ is obtained from the graph of $f(x)$.

6. Given $f(x) = \cos^{-1}(x)$ and $g(x) = -\cos^{-1}(x) - 1$

   (a) Graph both functions on the same coordinate system, using what you know about the domain of $\cos^{-1}(x)$ to help select an appropriate window.

   (b) Explain how the graph of $g(x)$ is obtained from the graph of $f(x)$.

7.5 Trigonometric Equations

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Exercises

1. Explain the difference between evaluating $\sin^{-1}(0.4567)$ and solving the equation $\sin(x) = 0.4567$.

2. Explain the difference between evaluating $\cos^{-1}(-0.1234)$ and solving the equation $\cos(x) = -0.1234$.

3. Determine at least 4 solutions in $[-2\pi, 2\pi]$, using 4 decimal places, for $\sin(x) + \cos(x) = \sin^2(x) + \cos^2(x)$.

4. Find all values of x for which $\sin(2 - x) \leq 1 - 0.5x$, using 4 decimal places.

5. Given $f(x) = \dfrac{1 - \cos(2x)}{x}$, do the following

   (a) Find the $x-$intercepts algebraically.

   (b) Graph $f(x) = \dfrac{1 - \cos(2x)}{x}$, showing at least 4 $x-$intercepts.
   Find one non-zero intercept on the calculator graph to verify your algebraic answer. (It should match to at least 3 - 4 decimal places.)

   (c) Is this function even, odd, or neither?

   (d) If the function is even or odd, verify this algebraically by plugging in $-x$ and simplifying.

6. Given $f(x) = \cos\left(\dfrac{x}{2}\right) - 1$, do the following

   (a) Find the $x-$intercepts algebraically.

   (b) Graph $f(x) = \cos\left(\dfrac{x}{2}\right) - 1$ showing at least 4 $x-$intercepts.
   Find one non-zero intercept on the calculator graph to verify your algebraic answer. (It should match to at least 3 - 4 decimal places.)

   (c) Is this function even, odd, or neither?

   (d) If the function is even or odd, verify this algebraically by plugging in $-x$ and simplifying.